

# ON THE STRUCTURE AND REPRESENTATIONS OF THE INSERTION-ELIMINATION LIE ALGEBRA

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**ABSTRACT.** We examine the structure of the insertion-elimination Lie algebra on rooted trees introduced in [CK]. It possesses a triangular structure  $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathbb{C}.d \oplus \mathfrak{n}_-$ , like the Heisenberg, Virasoro, and affine algebras. We show in particular that it is simple, which in turn implies that it has no finite-dimensional representations. We consider a category of lowest-weight representations, and show that irreducible representations are uniquely determined by a "lowest weight"  $\lambda \in \mathbb{C}$ . We show that each irreducible representation is a quotient of a Verma-type object, which is generically irreducible.

## 1. INTRODUCTION

The insertion-elimination Lie algebra  $\mathfrak{g}$  was introduced in [CK] as a means of encoding the combinatorics of inserting and collapsing subgraphs of Feynman graphs, and the ways the two operations interact. A more abstract and universal description of these two operations is given in terms of rooted trees, which encode the hierarchy of subdivergences within a given Feynman graph, and it is this description that we adopt in this paper. More precisely,  $\mathfrak{g}$  is generated by two sets of operators  $\{D_t^+\}$ , and  $\{D_t^-\}$ , where  $t$  runs over the set of all rooted trees, together with a grading operator  $d$ . In [CK]  $\mathfrak{g}$  was defined in terms of its action on a natural representation  $\mathbb{C}\{\mathbb{T}\}$ , where the latter denotes the vector space spanned by rooted trees. For  $s \in \mathbb{C}\{\mathbb{T}\}$ ,  $D_t^+.s$  is a linear combination of the trees obtained by attaching  $t$  to  $s$  in all possible ways, whereas  $D_t^-.s$  is a linear combination of all the trees obtained by pruning the tree  $t$  from branches of  $s$ .  $\mathfrak{n}_+ = \{D_t^+\}$  and  $\mathfrak{n}_- = \{D_t^-\}$  form two isomorphic nilpotent Lie subalgebras, and  $\mathfrak{g}$  has a triangular structure

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathbb{C}.d \oplus \mathfrak{n}_-$$

as well as a natural  $\mathbb{Z}$ -grading by the number of vertices of the tree  $t$ . The Hopf algebra  $U(\mathfrak{n}_\pm)$  is dual to Kreimer's Hopf algebra of rooted trees [K].

This note aims to establish a few basic facts regarding the structure and representation theory of  $\mathfrak{g}$ . We begin by showing that  $\mathfrak{g}$  is simple, which together with its infinite-dimensionality implies that it has no non-trivial finite-dimensional representations, and that any non-trivial representation is necessarily faithful. We then proceed to develop a highest-weight theory for  $\mathfrak{g}$  along the lines of [K1, K2].

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In particular, we show that every irreducible highest-weight representation of  $\mathfrak{g}$  is a quotient of a Verma-like module, and that these are generically irreducible.

One can define a larger, "two-parameter" version of the insertion-elimination Lie algebra  $\tilde{\mathfrak{g}}$ , where operators are labelled by pairs of trees  $D_{t_1, t_2}$  (roughly speaking, in acting on  $\mathbb{C}\{\mathbb{T}\}$ , this operator replaces occurrences of  $t_1$  by  $t_2$ ). In the special case of ladder trees,  $\tilde{\mathfrak{g}}$  was studied in [M, KM1, KM2]. The finite-dimensional representations of the nilpotent subalgebras  $\mathfrak{n}_{\pm}$  as well as many other aspects of the Hopf algebra  $U(\mathfrak{n}_{\pm})$  were studied in [F].

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## 2. THE INSERTION-ELIMINATION LIE ALGEBRA ON ROOTED TREES

In this section, we review the construction of the insertion-elimination Lie algebra introduced in [CK], with some of the notational conventions introduced in [M].

Let  $\mathbb{T}$  denote the set of rooted trees. An element  $t \in \mathbb{T}$  is a tree (finite, one-dimensional contractible simplicial complex), with a distinguished vertex  $r(t)$ , called the root of  $t$ . Let  $V(t)$  and  $E(t)$  denote the set of vertices and edges of  $t$ , and let

$$|t| = \#V(t)$$

Let  $\mathbb{C}\{\mathbb{T}\}$  denote the vector space spanned by rooted trees. It is naturally graded,

$$(2.1) \quad \mathbb{C}\{\mathbb{T}\} = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathbb{C}\{\mathbb{T}\}_n$$

where  $\mathbb{C}\{\mathbb{T}\}_n = \text{span}\{t \in \mathbb{T} \mid |t| = n\}$ .  $\mathbb{C}\{\mathbb{T}\}_0$  is spanned by the empty tree, which we denote by  $\mathbf{1}$ . We have

$$\begin{aligned} \mathbb{C}\{\mathbb{T}\}_0 &= \langle \mathbf{1} \rangle & \mathbb{C}\{\mathbb{T}\}_1 &= \langle \bullet \rangle & \mathbb{C}\{\mathbb{T}\}_2 &= \langle \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \rangle \\ \mathbb{C}\{\mathbb{T}\}_3 &= \langle \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array}, \begin{array}{c} \bullet & & \bullet \\ & \diagdown \quad \diagup \\ & \bullet \end{array} \rangle \end{aligned}$$

where  $\langle, \rangle$  denotes span, and the root is the vertex at the top. If  $e \in E(t)$ , by a *cut along e* we mean the operation of cutting  $e$  from  $t$ . This divides  $t$  into two components -  $R_e(t)$  containing the root, and  $P_e(t)$ , the remaining one.  $R_e(t)$  and  $P_e(t)$  are naturally rooted trees, with  $r(R_e(t)) = r(t)$  and  $r(P_e(t)) = (\text{endpoint of } e)$ . Note that  $V(t) = V(R_e(t)) \cup V(P_e(t))$ .

Let  $\mathfrak{g}$  denote the Lie algebra with generators  $D_t^+, D_t^-, d, t \in \mathbb{T}$ , and relations

$$(2.2) \quad [D_{t_1}^+, D_{t_2}^+] = \sum_{v \in V(t_2)} D_{t_2 \cup_v t_1}^+ - \sum_{v \in V(t_1)} D_{t_1 \cup_v t_2}^+$$

$$(2.3) \quad [D_{t_1}^-, D_{t_2}^-] = \sum_{v \in V(t_1)} D_{t_1 \cup_v t_2}^- - \sum_{v \in V(t_2)} D_{t_2 \cup_v t_1}^-$$

$$(2.4) \quad [D_{t_1}^-, D_{t_2}^+] = \sum_{t \in \mathbb{T}} \alpha(t_1, t_2; t) D_t^+ + \sum_{t \in T} \beta(t_1, t_2; t) D_t^-$$

$$(2.5) \quad [D_t^-, D_t^+] = d$$

$$(2.6) \quad [d, D_t^-] = -|t| D_t^-$$

$$(2.7) \quad [d, D_t^+] = |t| D_t^+$$

where for  $s, t \in \mathbb{T}$ , and  $v \in V(s)$   $s \cup_v t$  denotes the rooted tree obtained by joining the root of  $t$  to  $s$  at the vertex  $v$  via a single edge, and

- $\alpha(t_1, t_2; t) = \#\{e \in E(t_2) \mid R_e(t_2) = t, P_e(t_2) = t_1\}$
- $\beta(t_1, t_2; t) = \#\{e \in E(t_1) \mid R_e(t_1) = t, P_e(t_1) = t_2\}$

Thus, for example

$$\begin{aligned} [D_{\bullet}^+, D_{\bullet}^+] &= D_{\bullet}^+ + 2D_{\bullet}^+ - D_{\bullet}^+ \\ [D_{\bullet}^-, D_{\bullet}^-] &= -D_{\bullet}^- - 2D_{\bullet}^- + D_{\bullet}^- \\ [D_{\bullet}^-, D_{\bullet}^+] &= 2D_{\bullet}^+ \end{aligned}$$

$\mathfrak{g}$  acts naturally on  $\mathbb{C}\{\mathbb{T}\}$  as follows. If  $s \in \mathbb{T}$ , viewed as an element of  $\mathbb{C}\{\mathbb{T}\}$ , and  $t \in \mathbb{T}$ , then

$$D_t^+(s) = \sum_{v \in V(s)} s \cup_v t$$

$$D_t^-(s) = \sum_{e \in E(s), P_e(s)=t} R_e(s)$$

$$d(s) = |s|s$$

### 3. STRUCTURE OF $\mathfrak{g}$

Let  $\mathfrak{n}_+$  and  $\mathfrak{n}_-$  be the Lie subalgebras of  $\mathfrak{g}$  generated by  $D_t^+$  and  $D_t^-$ ,  $t \in \mathbb{T}$ . We have a triangular decomposition

$$(3.1) \quad \mathfrak{g} = \mathfrak{n}_+ \oplus \mathbb{C}.d \oplus \mathfrak{n}_-$$

The relations 2.5, 2.6, and 2.7 imply that for every  $t \in \mathbb{T}$

$$\mathfrak{g}^t = \langle D_t^+, D_t^-, d \rangle$$

forms a Lie subalgebra isomorphic to  $\mathfrak{sl}_2$ . We have that  $\mathfrak{g}_t \cap \mathfrak{g}_s = \mathbb{C}.d$  if  $s \neq t$ . Assigning degree  $|t|$  to  $D_t^+$ ,  $-|t|$  to  $D_t^-$ , and 0 to  $d$  equips  $\mathfrak{g}$  with a  $\mathbb{Z}$ -grading.

$$\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n$$

$\mathfrak{g}$  possesses an involution  $\iota$ , with

$$\iota(D_t^+) = D_t^-, \quad \iota(D_t^-) = D_t^+, \quad \iota(d) = -d$$

Thus  $\iota$  is a gradation-reversing Lie algebra automorphism exchanging  $\mathfrak{n}_+$  and  $\mathfrak{n}_-$ .

**Theorem 3.1.**  *$\mathfrak{g}$  is a simple Lie algebra*

*Proof.* Suppose that  $\mathcal{I} \subset \mathfrak{g}$  is a proper Lie ideal. If  $x \in \mathcal{I}$ , let  $x = \sum_i x_i$ ,  $x_i \in \mathfrak{g}_i$  be its decomposition into homogenous components. We have

$$[d, x] = \sum_n n x_n$$

which implies that  $x_n \in \mathcal{I}$  for every  $n$  (because the Vandermonde determinant is invertible) i.e.  $\mathcal{I} = \bigoplus_{n \in \mathbb{Z}} (\mathcal{I} \cap \mathfrak{g}_n)$ . Suppose now that  $x_n \in \mathfrak{g}_n$ ,  $n > 0$ . We can write  $x_n$  as a linear combination of  $n$ -vertex rooted trees

$$(3.2) \quad x_n = \sum_{t \in \mathbb{T}_n} \alpha_t \cdot t$$

We proceed to show that  $D_{\bullet}^+ \in \mathcal{I}$ , where  $\bullet$  is the rooted tree with one vertex. Let  $S(x_n) \subset \mathbb{T}_n$  be the subset of  $n$ -vertex trees occurring with a non-zero  $\alpha_t$  in 3.2. Given a rooted tree  $t$ , let  $St(t)$  denote the set of rooted trees obtained by removing all the edges emanating from the root. Let

$$St(x_n) = \bigcup_{s \in S(x_n)} St(s)$$

and let  $\xi \in St(x_n)$  be of maximal degree. It is easy to see that  $[D_{\xi}^-, x_n]$  is a non-zero element of  $\mathfrak{g}_{n-|\xi|}$ . Starting with  $x_n \in \mathfrak{n}_+$ ,  $x_n \neq 0$ , and repeating this process if necessary, we eventually obtain a non-zero element of  $\mathfrak{g}_1 = \langle D_{\bullet}^+ \rangle$ . Now,  $[D_{\bullet}^-, D_{\bullet}^+] = d$ , and since  $[d, \mathfrak{g}] = \mathfrak{g}$ , this implies  $\mathcal{I} = \mathfrak{g}$ . We have thus shown that if  $\mathcal{I}$  is proper, then

$$\mathcal{I} \cap \mathfrak{n}_+ = 0$$

Applying  $\iota$  shows that  $\mathcal{I} \cap \mathfrak{n}_- = 0$  as well, and it is clear that  $\mathcal{I} \cap \mathbb{C}.d = 0$ . □

We can now use this result to deduce a couple of facts about the representation theory of  $\mathfrak{g}$ .

**Corollary 3.1.** *If  $V$  is a non-trivial representation of  $\mathfrak{g}$ , then  $V$  is faithful.*

**Corollary 3.2.**  *$\mathfrak{g}$  has no non-trivial finite-dimensional representations.*

The latter can also be easily deduced by analyzing the action of the  $\mathfrak{sl}_2$  subalgebras  $\mathfrak{g}^t$  as follows. Suppose that  $V$  is a finite-dimensional representation of  $\mathfrak{g}$ . To show that  $V$  is trivial, it suffices to show that it restricts to a trivial representation of  $\mathfrak{g}^t$  for every  $t \in \mathbb{T}$ . This in turn, will follow if we can show that for a *single* tree  $t \in \mathbb{T}$ ,  $\mathfrak{g}^t$  acts trivially, because this implies that  $d$  acts trivially, and  $\mathbb{C}.d \subset \mathfrak{g}^t$  plays the role of the Cartan subalgebra. Let

$$V = \bigoplus_{i=1 \dots k} V_{\delta_i}$$

be a decomposition of  $V$  into  $d$ -eigenspaces - i.e. if  $v \in V_{\delta_i}$ , then  $d.v_i = \delta_i v$ . Since  $V$  is finite-dimensional, the set  $\{\delta_i\}$  is bounded, and so lies in a disc of radius  $R$  in  $\mathbb{C}$ . If  $v \in V_{\delta_i}$  then  $[d, D_t^+] = |t|D_t^+$  implies that  $D_t^+.v \in V_{\delta_i+|t|}$ . Choosing a  $t \in \mathbb{T}$  such that  $|t| > 2R$  shows that  $D_t^+.v = 0$  for every  $v \in V$ .

**3.1. Lowest-weight representations of  $\mathfrak{g}$ .** We begin by examining the "defining" representation  $\mathbb{C}\{\mathbb{T}\}$  of  $\mathfrak{g}$  introduced in section 2. Its decomposition into  $d$ -eigenspaces is given by 2.1. Given a representation  $V$  of  $\mathfrak{g}$  on which  $d$  is diagonalizable, with finite-dimensional eigenspaces, and writing

$$V = \bigoplus_{\delta} V_{\delta}$$

for this decomposition, we define the *emphcharacter* of  $V$ ,  $char(V, q)$  to be the formal series

$$char(V, q) = \sum_{\delta} dim(V_{\delta}) q^{\delta}$$

The case  $V = \mathbb{C}\{\mathbb{T}\}$ , where  $dim(V_n)$  is the number of rooted trees on  $n$  vertices, suggests that representations of  $\mathfrak{g}$  may contain interesting combinatorial information. The triangular structure 3.1 of  $\mathfrak{g}$  suggests that a theory of highest- or lowest-weight representations may be appropriate.

**Definition 3.1.** We say that a representation  $V$  of  $\mathfrak{g}$  is *lowest-weight* if the following properties hold

- (1)  $V = \bigoplus V_{\delta}$  is a direct sum of finite-dimensional eigenspaces for  $d$ .
- (2) The eigenvalues  $\delta$  are bounded in the sense that there exists  $L \in \mathbb{R}$  such that  $Re(\delta) \geq L$ .

We call the  $\delta$  the *weights* of the representation, and category of such representations  $\mathcal{O}$ . If  $V \in \mathcal{O}$ , we say  $v \in V_\delta$  is a *lowest-weight vector* if  $\mathfrak{n}_- v = 0$ . Since  $D_t^-$  decreases the weight of a vector by  $|t|$ , and the weights all lie in a half-plane, it is clear that every  $V \in \mathcal{O}$  contains a lowest-weight vector.

Recall that a representation  $V$  of  $\mathfrak{g}$  is *indecomposable* if it cannot be written as  $V = V_1 \oplus V_2$  for two non-zero representations. Let  $U(\mathfrak{h})$  denote the universal enveloping algebra of a Lie algebra  $\mathfrak{h}$ .

**Lemma 3.1.** *If  $v \in V_\lambda$  is a lowest-weight vector, then  $U(\mathfrak{n}_+).v$  is an indecomposable representation of  $\mathfrak{g}$*

*Proof.*  $U(\mathfrak{g}).v$  is clearly the smallest sub-representation of  $V$  containing  $v$ . The decomposition 3.1 together with the PBW theorem implies that

$$U(\mathfrak{g}) = U(\mathfrak{n}_+) \otimes \mathbb{C}[d] \otimes U(\mathfrak{n}_-)$$

Because  $v$  is a lowest-weight vector,  $\mathbb{C}[d] \otimes U(\mathfrak{n}_-).v = \mathbb{C}.v$ . It follows that  $U(\mathfrak{g}).v = U(\mathfrak{n}_+).v$ . That the latter is indecomposable follows from the fact that in  $U(\mathfrak{n}_+).v$ , the weight space corresponding to  $\lambda$  is one-dimensional, and so if  $U(\mathfrak{n}_+).v = V_1 \oplus V_2$ , then  $v \in V_1$  or  $v \in V_2$ .  $\square$

Observe that

$$U(\mathfrak{n}_+).v = \oplus (U(\mathfrak{n}_+).v)_{\lambda+k}, \quad k \in \mathbb{Z}_{\geq 0}$$

where  $(U(\mathfrak{n}_+).v)_{\lambda+k}$  is spanned by monomials of the form

$$(3.3) \quad D_{t_1}^+ D_{t_2}^+ \cdots D_{t_i}^+.v$$

with  $|t_1| + \cdots + |t_i| = k$ .

The category  $\mathcal{O}$  contains Verma-like modules. For  $\lambda \in \mathbb{C}$ , let  $\mathbb{C}_\lambda$  denote the one-dimensional representation of  $\mathbb{C}.d \oplus \mathfrak{n}_-$  on which  $\mathfrak{n}_-$  acts trivially, and  $d$  acts by multiplication by  $\lambda$ .

**Definition 3.2.** The  $\mathfrak{g}$ -module

$$W(\lambda) = U(\mathfrak{g}) \otimes_{\mathbb{C}[d] \otimes U(\mathfrak{n}_-)} \mathbb{C}_\lambda$$

will be called *the Verma module* of lowest weight  $\lambda$ .

Choosing an ordering on trees yields a PBW basis for  $\mathfrak{n}_+$ , and thus also a basis of the form 3.3 for  $W(\lambda)$ .

Given a representation  $V \in \mathcal{O}$ , and a lowest weight vector  $v \in V_\lambda$ , we obtain a map of representations

$$(3.4) \quad \begin{aligned} W(\lambda) &\mapsto V \\ \mathbf{1} &\mapsto v \end{aligned}$$

**Lemma 3.2.** *If  $V \in \mathcal{O}$  is an irreducible representation, then  $V$  is the quotient of a Verma module.*

*Proof.* Since  $V \in \mathcal{O}$ ,  $V$  possesses a lowest-weight vector  $v \in V_\lambda$  for some  $\lambda \in \mathbb{C}$ . Since  $V$  is irreducible,  $V = U(\mathfrak{g}).v = U(\mathfrak{n}_+).v$ . The latter is a quotient of  $W(\lambda)$ .  $\square$

We have

$$\begin{aligned} \text{Char}(W(\lambda)) &= q^\lambda \sum_{n \in \mathbb{Z}_{\geq 0}} \dim(\mathbb{C}\{\mathbb{T}\}_n) q^n \\ &= q^\lambda \prod_{n \in \mathbb{Z}_{\geq 0}} \frac{1}{(1 - q^n)^{P(n)}} \end{aligned}$$

where  $P(n)$  is the number of primitive elements of degree  $n$  in  $\mathcal{H}_K$ .

**3.2. Irreducibility of  $W(\lambda)$ .** It is a natural question whether  $W(\lambda)$  is irreducible. In this section we prove the following result:

**Theorem 3.2.** *For  $\lambda$  outside a countable subset of  $\mathbb{C}$  containing 0,  $W(\lambda)$  is irreducible.*

*Proof.* Let  $v \neq 0$  be a basis for  $W(\lambda)_\lambda$ .  $W(\lambda)$  contains a proper sub-representation if and only if it contains a lowest-weight vector  $w$  such that  $w \notin \mathbb{C}.v$ . In  $W(0)$ ,  $D_\bullet^+.v \in W(0)_1$  is a lowest-weight vector, since

$$D_\bullet^- D_\bullet^+.v = D_\bullet^+ D_\bullet^-.v + d.v = 0$$

and  $D_t^-.v = 0$  for all  $t \in \mathbb{T}$  with  $|t| \geq 2$  by degree considerations. It follows that  $W(0)$  is not irreducible.

If  $I = (t_1, \dots, t_k)$  is a  $k$ -tuple of trees such that

$$t_1 \preceq t_2 \preceq \dots \preceq t_k$$

in the chosen order, let  $D_I^+.v$  denote the vector

$$(3.5) \quad D_{t_k}^+ \dots D_{t_1}^+.v \in W(\lambda)$$

$w \in W(\lambda)_{\lambda+n}$  is a lowest-weight vector if and only if

$$(3.6) \quad D_t^-.w = 0$$

for all  $t$  such that  $|t| \leq n$ . Writing  $w$  in the basis 3.5

$$w = \sum_{|I|=n} \alpha_I D_I^+.v$$

the conditions 3.6 translate into a system of equations for the coefficients  $\alpha_I$ . For example, if  $w \in W(\lambda)_{\lambda+2}$ , then

$$w = \alpha_1 D_{\bullet}^+.v + \alpha_2 D_\bullet^+ D_\bullet^+.v$$

and conditions  $D_{\bullet}^-.v = 0$ ,  $D_{\bullet}^-.w = 0$  translate into

$$\begin{aligned}\lambda\alpha_1 + \lambda\alpha_2 &= 0 \\ \alpha_1 + (2\lambda + 1)\alpha_2 &= 0\end{aligned}$$

The determinant of the corresponding matrix is  $2\lambda^2$ , and so for  $\lambda \neq 0$ , there is no lowest-weight vector  $w \in W(\lambda)_{\lambda+2}$ . For a general  $n$ , the system can be written in the form

$$(A + \lambda B)[\alpha_I] = 0$$

where  $A$  and  $B$  are matrices whose entries are non-negative integers. Let

$$f_n(\lambda) = \dim(\text{Ker}(A + \lambda B))$$

Then for every  $r \in \mathbb{N}$

$$S_{n,r} = \{\lambda \in \mathbb{C} \mid f_n(\lambda) \geq r\}$$

if proper, is a finite subset of  $\mathbb{C}$ , since the condition is equivalent to the vanishing a finite collection of sub-determinants, each of which is a polynomial in  $\lambda$ . The set of  $\lambda \in \mathbb{C}$  for which  $W(\lambda)$  is irreducible is therefore

$$\bigcup_{n \in \mathbb{N}} \{\mathbb{C} \setminus S_{n,1}\}$$

The theorem will follow if  $S_{n,1}$  is proper for each  $n \in \mathbb{N}$ . This follows from the following Lemma.

□

**Lemma 3.3.**  *$Z(1)$  is irreducible.*

*Proof.* We begin by examining the representation  $\mathbb{C}\{\mathbb{T}\}$ . The degree 0 subspace  $\mathbb{C}.1$  is a trivial representation of  $\mathfrak{g}$ . Let  $M$  denote the quotient  $\mathbb{C}\{\mathbb{T}\}/\mathbb{C}.1$ . It is easily seen that the exact sequence

$$0 \mapsto \mathbb{C} \mapsto \mathbb{C}\{\mathbb{T}\} \mapsto M \mapsto 0$$

is non-split.  $M$  has highest weight 1, and the subspace  $M_1$  can be identified with the span of the tree on one vertex  $\bullet$ . By the universal property of Verma modules, 3.4 we have a map

$$(3.7) \quad W(1) \mapsto M$$

sending the lowest-weight vector of  $W(1)$  to  $\bullet$ . Now,  $W(1)_n$  is spanned by all vectors 3.5 such that  $|t_1| + \cdots + |t_k| = n - 1$ , and so can be identified with the set of forests on  $n - 1$  vertices, while  $M_n$  can be identified with  $\mathbb{C}\{\mathbb{T}\}_n$ . The operation of adding a root to a forest on  $n - 1$  vertices to produce a rooted tree with  $n$  vertices yields an isomorphism  $W(1)_n \cong M_n$ . Thus, if the map 3.7 is a surjection, it is an isomorphism. This in turn, follows from the fact that  $M$  is irreducible.



It suffices to show that  $M_n$  contains no lowest-weight vectors for  $n > 1$ . This follows from an argument similar to the one used to prove 3.1. Let  $w \in M_n$ , and write

$$w = \alpha_1 t_1 + \cdots \alpha_k t_k$$

where  $|t_i| = n$  and we may assume that  $\alpha_i \neq 0$ . In the notation of 3.1, let  $\xi \in St(w)$  be of maximal degree. Then

$$D_\xi^- . w \neq 0$$

Thus,  $M$  is irreducible, and hence isomorphic to  $W(1)$  by the map 3.7.  $\square$

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